

# A Riemann-Hilbert problem for biorthogonal polynomials

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## Abstract

We characterize the biorthogonal polynomials that appear in the theory of coupled random matrices via a Riemann-Hilbert problem. Our Riemann-Hilbert problem is different from the ones that were proposed recently by Ercolani and McLaughlin, Kapaev, and Bertola et al. We believe that our formulation may be tractable to asymptotic analysis.

## 1 Introduction

The biorthogonal polynomials that appear in the theory of coupled random matrices [3, 10, 11, 18] are characterized by the property that

$$\iint p_k(x)q_j(y)e^{-V(x)-W(y)+2\tau xy}dxdy = 0, \quad \text{if } j \neq k, \quad (1.1)$$

where  $p_k$  and  $q_j$  are polynomials of exact degrees  $k$  and  $j$ , respectively. In (1.1) we have that  $V, W : \mathbb{R} \rightarrow \mathbb{R}$  are given functions with sufficient increase at infinity so that the integrals converge, and  $\tau \neq 0$  is a nonzero coupling constant. The integration is over  $\mathbb{R}^2$ .

Ercolani and McLaughlin [10] showed that the two sequences of biorthogonal polynomials  $(p_k)$  and  $(q_j)$  exist, that they are unique, and moreover, that  $p_k$  has exactly  $k$  simple real zeros, see also [22]. They also gave a Riemann-Hilbert formulation for the biorthogonal polynomials which is non-local in character. Recently, for the case that  $V$  and  $W$  are polynomials, Kapaev [13] and Bertola et al. [4] gave local Riemann-Hilbert problems. If

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$d = \deg W$ , then the Riemann-Hilbert problems for  $p_k$  are formulated for  $d \times d$ -matrix valued functions in [4, 13].

In this note we derive a different Riemann-Hilbert problem. Our Riemann-Hilbert problem is based on the fact that the biorthogonal polynomials can be characterized as multiple orthogonal polynomials (see below). The formulation of a Riemann-Hilbert problem for multiple orthogonal polynomials is due to Van Assche et al. [21]

An outstanding problem in random matrix theory is to provide a rigorous asymptotic of eigenvalue statistics for coupled random matrices. The basic example is the so-called 2-matrix model in which we have a probability measure on pairs  $(M_1, M_2)$  of Hermitian  $N \times N$  matrices of the form

$$\frac{1}{Z_N} \exp(-\text{Tr}(V(M_1) + W(M_2) - 2\tau M_1 M_2)) dM_1 dM_2.$$

Statistical quantities on eigenvalues of  $M_1$  and  $M_2$  can be expressed in terms of the biorthogonal polynomials  $p_k$  and  $q_j$  given by (1.1), see [18, 11]. The connection to biorthogonal polynomials would be very useful, if one has, in addition, a complete asymptotic description of the biorthogonal polynomials. Then it would be possible to compute eigenvalue statistics in the large  $N$  limit. Indeed, although the calculations are somewhat involved, this has been carried out in the Gaussian case  $V(x) = x^2$ ,  $W(y) = ay^2$ , in [10]. Rigorous asymptotics for biorthogonal polynomials with more general functions  $V$  and  $W$  are not known.

In the 1-matrix case, the statistical quantities on eigenvalues are given in terms of orthogonal polynomials [17], which have been characterized by a Riemann-Hilbert problem [12]. The steepest descent / stationary phase method for Riemann-Hilbert problems was applied with great success to orthogonal polynomials [8, 9, 14]. As a result, the large  $N$  asymptotics of 1-matrix models could be carried out in great detail, which in particular provided a proof of the universality of eigenvalue spacings for a large class of matrix models [5, 8, 16]. So there is hope that a similar asymptotic analysis of a Riemann-Hilbert problem for biorthogonal polynomials will lead to large  $N$  asymptotics for 2-matrix models. The formulation of a suitable Riemann-Hilbert problem is only a first step in this direction.

For simplicity and clarity we formulate and prove our Riemann-Hilbert problem for the biorthogonal polynomial  $p_k$  for the first non-trivial case. This is the case where  $W$  is a polynomial of degree 4. Indeed, if  $W$  is a polynomial of degree 2, say  $W(y) = y^2 + 2by + c$ , then  $p_k$  is the orthogonal polynomial with respect to the weight  $e^{-V(x) + (\tau x - b)^2}$  on  $\mathbb{R}$  and so there is a Riemann-Hilbert problem for  $p_k$  [7, 12].

Thus we assume that  $W$  is a polynomial of degree 4 and we define

$$w_j(x) = \int y^j e^{-V(x)-W(y)+2\tau xy} dy, \quad j = 0, 1, 2. \quad (1.2)$$

These functions appear in the formulation of our Riemann-Hilbert problem.

### Riemann-Hilbert problem for $Y$

The problem is to find a  $4 \times 4$  matrix valued function  $Y : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{4 \times 4}$  having the following three properties.

- (a)  $Y$  is analytic on  $\mathbb{C} \setminus \mathbb{R}$ .
- (b)  $Y$  has boundary values on  $\mathbb{R}$ , denoted by  $Y_+$  and  $Y_-$ , so that

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_0(x) & w_1(x) & w_2(x) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R}. \quad (1.3)$$

- (c) As  $z \rightarrow \infty$ , we have

$$Y(z) = \left( I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^k & 0 & 0 & 0 \\ 0 & z^{-n_0} & 0 & 0 \\ 0 & 0 & z^{-n_1} & 0 \\ 0 & 0 & 0 & z^{-n_2} \end{pmatrix}, \quad (1.4)$$

where  $k \in \mathbb{N}_0$ ,  $n_0 = \lfloor \frac{k+2}{3} \rfloor$ ,  $n_1 = \lfloor \frac{k+1}{3} \rfloor$ , and  $n_2 = \lfloor \frac{k}{3} \rfloor$ . Here  $\lfloor \cdot \rfloor$  denotes the integer part. (Note that  $k = n_0 + n_1 + n_2$ .)

The main result of this paper is that the Riemann-Hilbert problem for  $Y$  has a unique solution and that its  $(1, 1)$  entry  $Y_{11}$  is equal to the monic biorthogonal polynomial  $p_k$ . In what follows we use  $C(f)$  defined by

$$C(f)(z) = \frac{1}{2\pi i} \int \frac{f(x)}{x - z} dx, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

to denote the Cauchy transform of a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

**Theorem 1.1.** *Let the functions  $w_j$ ,  $j = 0, 1, 2$ , be given by (1.2) and let  $k \in \mathbb{N}_0$ . Then the above Riemann-Hilbert problem for  $Y$  has a unique*

solution given by

$$Y = \begin{pmatrix} p_k & C(p_k w_0) & C(p_k w_1) & C(p_k w_2) \\ p_{k-1}^{(0)} & C(p_{k-1}^{(0)} w_0) & C(p_{k-1}^{(0)} w_1) & C(p_{k-1}^{(0)} w_2) \\ p_{k-1}^{(1)} & C(p_{k-1}^{(1)} w_0) & C(p_{k-1}^{(1)} w_1) & C(p_{k-1}^{(1)} w_2) \\ p_{k-1}^{(2)} & C(p_{k-1}^{(2)} w_0) & C(p_{k-1}^{(2)} w_1) & C(p_{k-1}^{(2)} w_2) \end{pmatrix} \quad (1.5)$$

where  $p_k$  is the monic polynomial of degree  $k$  satisfying (1.1), and  $p_{k-1}^{(j)}$ ,  $j = 0, 1, 2$ , are three polynomials of degrees  $\leq k-1$ .

**Remark 1.2.** There is an immediate extension to polynomials  $W$  of arbitrary degree. If  $W$  is a polynomial of degree  $d$ , then the Riemann-Hilbert problem is for  $Y : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{d \times d}$  so that

$$Y \text{ is analytic on } \mathbb{C} \setminus \mathbb{R}. \quad (1.6)$$

The jump condition uses the  $d-1$  functions  $w_j(x) = \int y^j e^{-V(x)-W(y)+2\tau xy} dy$ ,  $j = 0, 1, \dots, d-2$ , and is given by

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_0(x) & w_1(x) & \cdots & \cdots & w_{d-3}(x) & w_{d-2}(x) \\ 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & & & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \quad (1.7)$$

for  $x \in \mathbb{R}$ . The asymptotic condition is

$$Y(z) = \left( I + O\left(\frac{1}{z}\right) \right) \text{diag} \left( z^k, z^{-n_0}, z^{-n_1}, \dots, z^{-n_{d-2}} \right) \quad (1.8)$$

where  $n_j = \left\lfloor \frac{k+d-2-j}{d-1} \right\rfloor$  for  $j = 0, 1, \dots, d-2$ , and  $\text{diag}(\cdot)$  denotes a diagonal matrix.

The Riemann-Hilbert problem (1.6)–(1.8) has a unique solution and  $Y_{11} = p_k$ , where  $p_k$  is the monic biorthogonal polynomial of degree  $k$ . The proof of the general case follows along the same lines as the proof of the case  $\deg W = 4$  that we will present in Sections 2 and 3 below.

**Remark 1.3.** By symmetry, there is a similar Riemann-Hilbert problem that characterizes the other biorthogonal polynomial  $q_j$ , in the case that  $V$  is a polynomial.

## 2 Multiple orthogonality

We assume that  $W$  is a polynomial of degree 4. In this section we will characterize the monic biorthogonal polynomial  $p_k$  of degree  $k$  through a set of orthogonality relations with respect to the three functions  $w_j$ ,  $j = 0, 1, 2$ . As in Theorem 1.1, we use  $n_0 = \lceil \frac{k+2}{3} \rceil$ ,  $n_1 = \lceil \frac{k+1}{3} \rceil$ , and  $n_2 = \lceil \frac{k}{3} \rceil$ .

**Lemma 2.1.** *We have*

$$\int p_k(x) x^i w_j(x) dx = 0, \quad \text{for } i = 0, 1, \dots, n_j - 1, \quad j = 0, 1, 2, \quad (2.1)$$

and these relations characterize the biorthogonal polynomial  $p_k$  among all monic polynomials of degree  $k$ .

*Proof.* Since  $W$  is a polynomial of degree 4, it is easy to see that

$$\frac{d^i}{dy^i} \left( y^j e^{-W(y)} \right) = \pi_{3i+j}(y) e^{-W(y)}$$

where  $\pi_{3i+j}$  is a polynomial of exact degree  $3i + j$ . For any function  $f$ , we then have if we integrate by parts  $i$  times

$$\begin{aligned} & \iint f(x) \pi_{3i+j}(y) e^{-V(x)-W(y)+2\tau xy} dx dy \\ &= \int f(x) e^{-V(x)} \int \frac{d^i}{dy^i} \left( y^j e^{-W(y)} \right) e^{2\tau xy} dy dx \\ &= (-1)^i \int f(x) e^{-V(x)} \int y^j e^{-W(y)} \frac{d^i}{dy^i} \left( e^{2\tau xy} \right) dy dx \\ &= (-2\tau)^i \int f(x) x^i e^{-V(x)} \int y^j e^{-W(y)+2\tau xy} dy dx \\ &= (-2\tau)^i \int f(x) x^i w_j(x) dx. \end{aligned} \quad (2.2)$$

If  $f$  is the biorthogonal polynomial  $p_k$  then the left-hand side of (2.2) is zero if  $3i + j < k$ . This corresponds exactly with  $i \leq n_j - 1$  for  $j = 0, 1, 2$ , so that by the right-hand side we have the relations (2.1).

Conversely, if  $f$  is a monic polynomial of degree  $k$  that satisfies the relations  $\int f(x) x^i w_j(x) dx = 0$  for  $i = 0, \dots, n_j - 1$ ,  $j = 0, 1, 2$ , then the

left-hand side of (2.2) is zero for  $i \leq n_j - 1$  and  $j = 0, 1, 2$ . The polynomials  $\pi_{3i+j}$  with  $i \leq n_j - 1$  and  $j = 0, 1, 2$  are a basis for the polynomials of degree  $\leq k - 1$ . Hence  $f$  is the biorthogonal polynomial  $p_k$ .  $\square$

**Remark 2.2.** The relations (2.1) are called multiple orthogonality relations of type II, see [1, 2, 19, 20, 21] for more on this subject.

### 3 Proof of Theorem 1.1

*Proof.* We first establish uniqueness in the standard way. The only thing we have to observe is that both the jump matrix in (1.3) and the diagonal matrix in the right-hand side of (1.4) have determinant one. Then the proof of uniqueness follows as in [7, Section 3.2].

We now prove that  $Y$  given by (1.5) satisfies the Riemann-Hilbert problem. First we consider the first row of  $Y$ . The conditions (1.3) and (1.4) give for the  $(1, 1)$  entry

$$Y_{11,+} = Y_{11,-}, \quad \text{and} \quad Y_{11}(z) = z^k + O(z^{k-1}),$$

These conditions are clearly satisfied if  $Y_{11} = p_k$ , since  $p_k$  is a monic polynomial of degree  $k$ .

For the other entries in the first row, the jump condition (1.3) then is

$$Y_{1j,+} = Y_{1j,-} + Y_{11,-} w_{j-2} = Y_{1j,-} + p_k w_{j-2} \quad j = 2, 3, 4.$$

By the Sokhotskii-Plemelj formula, this is satisfied by  $Y_{1j} = C(p_k w_{j-2})$ . The asymptotic condition (1.4) is

$$Y_{1j}(z) = O(z^{-n_{j-2}-1}) \quad \text{as } z \rightarrow \infty \quad (3.1)$$

and we have to check that this is satisfied for  $Y_{1j} = C(p_k w_{j-2})$ . If we use

$$\frac{1}{x-z} = - \sum_{i=0}^{n_{j-2}-1} \frac{x^i}{z^{i+1}} + \frac{x^{n_{j-2}}}{z^{n_{j-2}}} \frac{1}{x-z}$$

then we see that for  $j = 2, 3, 4$ ,

$$\begin{aligned} C(p_k w_{j-2})(z) &= \frac{1}{2\pi i} \int \frac{p_k(x) w_{j-2}(x)}{x-z} \\ &= - \sum_{i=0}^{n_{j-2}-1} \left( \frac{1}{2\pi i} \int p_k(x) x^i w_{j-2}(x) dx \right) z^{-i-1} \\ &\quad + \left( \frac{1}{2\pi i} \int \frac{p_k(x) w_{j-2}(x) x^{n_{j-2}}}{x-z} dx \right) z^{-n_{j-2}}. \end{aligned} \quad (3.2)$$

Because of the multiple orthogonal relations (2.1) satisfied by  $p_k$ , (3.2) reduces to

$$C(p_k w_{j-2})(z) = \left( \frac{1}{2\pi i} \int \frac{p_k(x) w_{j-2}(x) x^{n_{j-2}}}{x-z} dx \right) z^{-n_{j-2}}$$

which shows that (3.1) is indeed satisfied if  $Y_{1j} = C(p_k w_{j-2})$

Next we consider the second row of  $Y$ . The conditions  $Y_{21,+} = Y_{21,-}$  and  $Y_{21}(z) = O(z^{k-1})$  as  $z \rightarrow \infty$  are clearly satisfied if  $Y_{21}$  is a polynomial  $p_{k-1}^{(0)}$  of degree  $\leq k-1$ . The jump conditions for the other entries in the second row

$$Y_{2j,+} = Y_{2j,-} + Y_{1j,-} w_{j-2} = Y_{2j,-} + p_{k-1}^{(0)} w_{j-2}$$

are then also satisfied if  $Y_{2j} = C(p_{k-1}^{(0)} w_{j-2})$  for  $j = 2, 3, 4$ . We need to be able to choose  $p_{k-1}^{(0)}$  so that the asymptotic condition (1.3) is also satisfied, which means that

$$\begin{cases} C(p_{k-1}^{(0)} w_0) = z^{-n_0} + O(z^{-n_0-1}), \\ C(p_{k-1}^{(0)} w_j) = O(z^{-n_j-1}) \end{cases} \quad \text{for } j = 1, 2. \quad (3.3)$$

Expanding  $C(p_{k-1}^{(0)} w_j)$  as in (3.2), we see that (3.3) is satisfied if  $p_{k-1}^{(0)}$  is such that

$$\int p_{k-1}^{(0)}(x) x^i w_0(x) dx = 0, \quad \text{for } i = 0, \dots, n_0 - 2, \quad (3.4)$$

$$\int p_{k-1}^{(0)}(x) x^i w_0(x) dx = -2\pi i, \quad \text{for } i = n_0 - 1, \quad (3.5)$$

$$\int p_{k-1}^{(0)}(x) x^i w_j(x) dx = 0, \quad \text{for } i = 0, \dots, n_j - 1, j = 1, 2. \quad (3.6)$$

The conditions (3.4) and (3.6) give  $k-1$  homogeneous conditions on the  $k$  free coefficients of  $p_{k-1}^{(0)}$ , and so there exists a non-zero polynomial  $p_{k-1}^{(0)}$  satisfying these conditions. To be able to have (3.5) as well we must exclude the possibility that

$$\int p_{k-1}^{(0)}(x) x^{n_0-1} w_0(x) dx = 0. \quad (3.7)$$

However, if (3.7) would hold, then  $p_k + p_{k-1}^{(0)}$  would be a monic polynomial of degree  $k$  that satisfies the multiple orthogonality relations (2.1), which is impossible, since these relations characterize the biorthogonal polynomial

$p_k$  by Lemma 2.1. Thus (3.7) cannot hold. Then we can normalize  $p_{k-1}^{(0)}$  by multiplying it with a suitable constant, so that (3.5) is satisfied. This proves that we can indeed choose  $p_{k-1}^{(0)}$  so that the second row of  $Y$  satisfies all the conditions imposed by the Riemann-Hilbert problem.

In exactly the same way, we handle the third and fourth rows of  $Y$ .

This completes the proof of Theorem 1.1.  $\square$

## 4 Conclusion

We have characterized the biorthogonal polynomials that appear in the theory of coupled random matrices via a Riemann-Hilbert problem which is different from the Riemann-Hilbert problems derived in [4, 13]. Recent experience [6, 15] with similar higher order Riemann-Hilbert problems leads us to believe that our Riemann-Hilbert problem may be tractable to asymptotic analysis. However, up to now we have not been able to apply the steepest descent method successfully to this problem, and the actual asymptotic analysis of biorthogonal polynomials remains a major open problem.

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